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Absolute Convergence of Fourier Series of Convolution Functions

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1. INTRODUCTION

1.1. We shall consider functions integrable on $(0, 2\pi)$ and periodic with period 2π. Then the following theorem is known:

THEOREM I. *Let f be a continuous function. If there are two squarely integrable functions g and h such that*

$$f(x) = \frac{1}{\pi}\int_0^{2\pi} g(x+t)\,h(t)\,dt, \tag{1}$$

then the Fourier series of f converges absolutely. The converse holds also.

This theorem is due to Riesz ([*1*], I, p. 251 and [*2*], II, p. 184) and Chen [*3*]. The integral in (1) is called the convolution of g and h, and is denoted by

$$f(x) = (g * h)(x).$$

We shall ask whether we can make the condition for g weaker and the condition for h stronger in the first part of Theorem I.

We shall introduce a subclass of L^p $(p \geqq 1)$, defined by Hardy and Littlewood ([*1*] and [*2*]). If a function $g \in L^p$ satisfies the condition, for an a $(0 < a \leqq 1)$,

$$\exists A : \left(\int_0^{2\pi} |g(x+t) - g(x)|^p\,dx\right)^{1/p} \leqq A|t|^a \qquad \text{as } t \to 0, \tag{2}$$

then we say that g belongs to the class $\mathrm{Lip}(a,p)$. Evidently, $\mathrm{Lip}(a,p) \subset L^p$ and the class $\mathrm{Lip}(a,p)$ becomes larger when a or p decreases.

Chen [*4*] has proved the following:

THEOREM II. *If $g \in \mathrm{Lip}(a,p)$ and $h \in \mathrm{Lip}(b,q)$ with*

$$1 < p < 2, \qquad q > 1 \qquad \textit{and} \qquad a > b = 1/2p,$$

then the function $f = g * h$ has an absolutely convergent Fourier series.

Further, Yadav [*5*] proved

THEOREM III. *If $g \in Lip(a,p)$ and $h \in Lip(b,q)$ with*

$$1 < p < 2, \qquad 1/p + 1/q = 1 \qquad \text{and} \qquad a + b > 1/p,$$

then the function $f = g * h$ has an absolutely convergent Fourier series.

In these theorems neither the condition for g nor that for h is weaker than square integrability and both of a and b cannot become small when p and q approach 2.

1.2. We prove the following theorems:

THEOREM 1. *Let $1 < p < 2$ and $1/p + 1/q = 1$. If $g \in L^p$ and $h \in L^p$ and if, further,*

$$\int_0^1 \frac{(\omega_p(t;h))^q}{t^{q-1}} dt < \infty \tag{3}$$

where $\omega_p(t;h)$ denotes the L^p,modulus of continuity of the function h, defined by

$$\omega_p(t;h) = \sup_{0<u\leq t} \left(\int_0^{2\pi} |h(x+u) - h(x)|^p dx\right)^{1/p}, \tag{4}$$

then the function $f = g * h$ has an absolutely convergent Fourier series.

If $h \in \mathrm{Lip}(a,p)$, then, by (2) and (4),

$$\omega_p(t;h) = O(t^a) \qquad \text{as} \qquad t \to 0.$$

If $a > (2-p)/p$, then condition (3) is satisfied. Thus we get

COROLLARY 1. *Let $1 < p < 2$. If $g \in L^p$ and $h \in Lip(a,p)$ with $a > (2-p)/p$, then the function $f = g * h$ has an absolutely convergent Fourier series.*

In this corollary, if p is near 1, then $(2-p)/p$ is also near 1, and then a must also be near 1. If p is near 2, then $(2-p)/p$ is near zero and a can also be taken near zero.

In Theorem 1, we take $g = h$ and suppose that they satisfy condition (3). Then Theorem 1 gives

COROLLARY 2. *A function $h \in L^p$ $(1 < p < 2)$, satisfying condition* (3), *is in L^2.*

This shows that the condition for h in Theorem 1 is stronger than square integrability. This is quite natural. Combining Corollaries 1 and 2, we see that $\mathrm{Lip}(a,p) \subset L^2$ for $1 < p < 2$ and $a > (2-p)/p$. This is a special case of a theorem of Hardy and Littlewood [*8*].

THEOREM 2. *Theorem* 1 *need not be true when the integral* (3) *diverges. In particular, if $a = (2-p)/p$, then Corollary* 1 *does not hold in general.*

THEOREM 3. *In Corollary* 1, *the class $Lip(a,p)$ of h cannot be replaced by any L^s $(s > 2)$. That is, for any p, $1 < p < 2$, and any $s > 2$, there are $g \in L^p$ and $h \in L^s$ such that the Fourier series of $f = g * h$ does not converge absolutely.*

Let us now consider the limiting cases $p \to 1$ and $p \to 2$ in Corollary 1. If $p \to 1$, then the assumptions on g and h become

$$g \in L^1 \qquad \text{and} \qquad h \in \mathrm{Lip}\,(1, 1).$$

It is known that Lip (1, 1) is identical with BV (the class of functions of bounded variation). These conditions are not sufficient for absolute convergence of the Fourier series of $g * h$. On the other hand, if $p \to 2$ in Corollary 1, then the assumptions become

$$g \in L^2 \qquad \text{and} \qquad h \in \lim_{a \to 0} \mathrm{Lip}\,(a, 2).$$

The last class is a proper subclass of L^2 and so this case is a particular case of Theorem I.

THEOREM 4. *Let $1 < p < 2$ and $c > 0$. If $g \in L^p$ and $h \in L^p$ satisfy the conditions*

$$\sum_{n=-\infty}^{\infty} \frac{|c_n(g)|^p}{\log\,(|n|+2)} < \infty \tag{5}$$

where $c_n(g)$ is the nth *(complex) Fourier coefficient of the function g, and*[1]

$$\omega_p\,(t; h) \leqq A \Big/ \left(\log \frac{1}{t}\right)^{1+c}, \tag{6}$$

then the Fourier series of $f = g * h$ converges absolutely.

1.3. Theorem 1, 4 and III are special cases of the following key theorem:

THEOREM 5. *Let $1 < p < 2$, $1/p + 1/q = 1$, and let $\lambda(t)$ be a positive monotone (increasing or decreasing) function for $t > 0$ such that*

$$\exists A'' > A' > 0 : A'' > \lambda(t)/\lambda(2t) > A' \qquad \textit{for all } t > 0. \tag{7}$$

If $g \in L^p$ and $h \in L^p$ satisfy the conditions

$$\sum_{n=1}^{\infty} |c_n(g)|^p\,(\lambda(n))^p < \infty \tag{8}$$

and

$$\int_0^1 \frac{(\omega_p(t; h))^q}{t(\lambda(1/t))^q}\,dt < \infty, \tag{9}$$

then the function $f = g * h$ has an absolutely convergent Fourier series.

For the proof of this theorem, we use the following lemma due to Leindler [7] (cf. [6]).

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[1] A is used to denote an absolute constant which is different in different occurrences.

LEMMA. *Let* $1 < p < 2$ *and* $1/p + 1/q = 1$. *If* $f \in L^p$, *then*

$$\sum_{n=1}^{\infty} \frac{1}{\mu(n)} \sum_{m=n}^{\infty} |c_m(f)|^q \leqq A \int_0^1 \frac{dt}{t^2 \mu(1/t)} \left(\int_0^{2\pi} |f(x+t) - f(x-t)|^p \, dx \right)^{q/p}, \quad (10)$$

where

(i) $\mu(t)$ *is defined for* $t > 0$, *positive, monotone (increasing or decreasing) and satisfies condition* (7), *or more generally,*

(ii) $\mu(t)$ *is positive for* $t > 0$ *and*

$$\exists\, A'' > A' > 0 : A' \mu(2^{k-1}) < \mu(t) < A'' \mu(2^k) \quad (11)$$

for all t in the interval $(2^{k-1}, 2^k)$ *and for all* $k = 1, 2, \ldots$.

The case (i) is proved in [7] and more simply in [6]. The case (ii) is not stated explicitly in [6], but the proof given there still applies. A useful special case of (ii) is that

(iii) there are $\lambda_1(t)$ and $\lambda_2(t)$ defined for $t > 0$ such that $\mu(t) = \lambda_1(t)\, \lambda_2(t)$, $\lambda_1(t)$ is monotone increasing, $\lambda_2(t)$ is monotone decreasing and both of them satisfy condition (7).

2. Proof of Theorem 5. By (1), we have

$$c_n(f) = c_n(g) \cdot c_n(h) \qquad \text{for all } n.$$

Without loss of generality, we can suppose that $c_n(g)$ and $c_n(h)$ vanish for all negative n. By Hölder's inequality,

$$\sum_{n=1}^{\infty} |c_n(f)| \leqq \left(\sum_{n=1}^{\infty} |c_n(g)\lambda(n)|^p \right)^{1/p} \left(\sum_{n=1}^{\infty} |c_n(h)/\lambda(n)|^q \right)^{1/q}.$$

Since the first factor is finite by assumption (8), it is sufficient to prove that the second factor on the right side is finite. By condition (7),

$$\sum_{m=1}^{\infty} |c_n(h)|^q (\lambda(n))^{-q} = \sum_{k=1}^{\infty} \sum_{n=2^{k-1}}^{2^k-1} |c_n(h)|^q (\lambda(n))^{-q}$$

$$\leqq A \sum_{k=1}^{\infty} \frac{1}{(\lambda(2^k))^q} \sum_{n=2^{k-1}}^{2k-1} |c_n(h)|^q \leqq A \sum_{k=1}^{\infty} \frac{1}{(\lambda(2^k))^q} \sum_{n=2^{k-1}}^{\infty} |c_n(h)|^q$$

$$\leqq A \sum_{n=1}^{\infty} \frac{1}{n(\lambda(n))^q} \sum_{m=n}^{\infty} |c_m(h)|^q.$$

Now we want to use the lemma, taking $\mu(t) = t(\lambda(t))^q$. If $\lambda(t)$ is increasing, then so is $t(\lambda(t))^q$ and then condition (i) of the lemma is applicable. But if $\lambda(t)$ decreases, then condition (iii) holds by (7). Therefore the last sum is

$$\leqq A \int_0^1 \frac{dt}{t(\lambda(1/t))^q} \left(\int_0^{2\pi} |h(x+t) - h(x-t)|^p \, dx \right)^{p/q}.$$

By (4) we get

$$\sum_{n=1}^{\infty} |c_n(h)|^q (\lambda(n))^{-q} \leqq A \int_0^1 \frac{\omega^p(t;h)^q}{t(\lambda(1/t))^q}\, dt$$

where the right-side integral is finite by condition (9). This proves Theorem 5.

3. Proof of Theorems 1, 4 and III.

3.1. For the proof of Theorem 1, we use the following lemma due to Hardy and Littlewood ([*1*], II, p. 109).

LEMMA. *If* $g \in L^p$ $(1 < p \leqq 2)$, *then*

$$\sum_{n=1}^{\infty} |c_n(g)|^p n^{p-2} \leqq A \int_0^{2\pi} |g(x)|^p\, dx.$$

We take $\lambda(t) = t^{1-2/p}$ in Theorem 5, then condition (7) holds. Condition (8) follows from $g \in L^p$ and the lemma. Since

$$t\lambda(1/t)^q = t^{q-1},$$

condition (9) becomes condition (3). Thus we get Theorem 1 as a special case of Theorem 5.

3.2. In order to prove Theorem 4, we take $\lambda(t) = \log^{-1/p}(1/t + 2)$ $(t > 0)$, then condition (7) is satisfied. Condition (8) reduces to condition (5). If we assume (6), then

$$\int_0^1 \frac{(\omega_p(t;h))^q}{t(\lambda(1/t))^q}\, dt < A \int_0^1 \frac{dt}{t(\log(1/t))^{1+cq}} < \infty.$$

Hence condition (9) of Theorem 5 is satisfied. Thus Theorem 4 is a corollary of Theorem 5.

3.3. We shall derive Theorem III from Theorem 5. In the case $a \geqq 1/p$, $\sum |c_n(g)| < \infty$ and then $\sum |c_n(f)| < \infty$. Hence the Fourier series of $f * g$ converges absolutely. In the contrary case, we take $\lambda(t) = t^{-s}$ for $s > (1 - ap)/p$. Since $g \in \mathrm{Lip}(a,p)$ implies

$$c_n(g) = O(1/n^a),$$

we have

$$\sum |c_n(g)|^p (\lambda(n))^p \leqq A \sum \frac{1}{n^{ap+sp}} < \infty.$$

Thus condition (8) of Theorem 5 is satisfied. The integral of (9) is

$$\int_0^1 \frac{(\omega_p(t;h))^q}{t(\lambda(1/t))^q}\, dt < A \int_0^1 \frac{t^{bq}}{t^{1+sq}}\, dt$$

which is finite when $1 + sq - bq < 1$, i.e., $s < b$. An s with this property can be selected if $a + b > 1/p$.

4. Proof of Theorems 2 and 3.

4.1. For the proof of Theorem 2, we consider the function

$$h(t) = |t|^{-r} \qquad \text{for } |t| < \pi, \tag{11}$$
$$= h(t + 2\pi) \qquad \text{for all } t.$$

Then $h \in \mathrm{Lip}(a,p)$ for $a = (1/p) - r$ and $c_n(h)$ is exactly of order $|n|^{r-1}$ as $n \to \infty$ ([*1*], I, p. 190). Suppose that $a = (2-p)/p$, that is, $r = 1/q$, then

$$|c_n(h)| \cong A|n|^{r-1} = A|n|^{-1/p}.$$

Now we use the following lemma due to Hardy and Littlewood ([*1*], II, p. 129).

LEMMA. *Suppose that $c_n(g) = 0$ for $n < 0$ and $c_n(g)$ decreases monotonically to zero as $n \to \infty$. Then $g \in L^p$ if and only if*

$$\sum [c_n(g)]^p n^{p-2} < \infty.$$

By this lemma, there is a function $g \in L^p$ such that

$$c_n(g) = 1/n^{1/q} \log(n+1) \qquad \text{for } n > 0,$$
$$= 0 \qquad \text{otherwise.} \tag{12}$$

For the functions g and h defined by (11) and (12), we have

$$\sum_{n=1}^{\infty} |c_n(f)| = \sum_{n=1}^{\infty} |c_n(g)| \cdot |c_n(h)|$$

$$\cong A \sum_{n=1}^{\infty} \frac{1}{n^{1/q} \log(n+1)} \cdot \frac{1}{n^{1/p}} = A \sum_{n=1}^{\infty} \frac{1}{n \log(n+1)} = \infty.$$

Thus the Fourier series of $f = g * h$ does not converge absolutely. This proves the second part of Theorem 2. The first part is now also evident, since integral (3) diverges for h defined by (11) with $r = 1/q$.

4.2. The following is known ([*1*], I, p. 215):

LEMMA. *There is a sequence (ϵ_n) of signs such that the series*

$$\sum_{n=1}^{\infty} \frac{\epsilon_n e^{inx}}{\sqrt{n} \log(n+1)} \tag{13}$$

belongs to every L^s $(s > 0)$.

We define g by (12) and h as the sum (13). Then

$$\sum |c_n(f)| = \sum |c_n(g)| \cdot |c_n(h)| = \sum \frac{1}{n^{1/q} \log (n+1)} \cdot \frac{1}{\sqrt{n} \log (n+1)}$$

$$= \sum \frac{1}{n^{(1/2)+(1/q)} (\log (n+1))^2} = \infty,$$

since $1/2 + 1/q < 1$. Thus we get Theorem 3.

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